M—IDEALS IN COMPLEX FUNCTION SPACES AND ALGEBRAS*

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ABSTRACT

Convexity arguments are applied to characterize the M-ideals of a given complex function space $A \subseteq C(X)$. The main result is the following: A closed subspace J of A is an M-ideal if and only if $J = \{ a \in A \mid a \equiv 0 \text{ on } E\}$, where E is an M-set of X. Specializing to uniform algebras it is shown that M-ideals coincides with the algebraic ideals generated by p-sets, which in turn yields a description of the primitive ideal space, Prim A, as the Choquet-boundary endowed with p-set topology.

Introduction

The aim of this paper is to give a characterization of the M-ideals of a complex function space $A \subseteq \mathscr{C}_{\mathbb{C}}(X)$.

The concept of an M-ideal was defined for real Banach spaces by Alfsen and Effros [2], but it can be easily transferred to the complex case (Theorem 1.3).

The main result is the following: Let J be a closed subspace of a complex function space A. Then J is an M-ideal in A if and only if

$$J = \{a \in A \mid a \equiv 0 \text{ on } E\},$$

where $E \subseteq X$ is an A-convex set having the properties:

(i) $\mu \in M_1^+(\partial_A X)$, $\nu \in M_1^+(E)$, $\mu - \nu \in A^\perp \Rightarrow \mathrm{Supp}(\mu) \subseteq E$

(ii) $\mu \in A^\perp \cap (\partial_A X) \Rightarrow \mu|_E \in A^\perp$.

In case A is a uniform algebra these sets are precisely the p-sets (generalized peak sets).

* This work was in part supported by the Danish Science Council J. no. 511–1105.

Received November 3, 1971.

Following the lines of [2], we shall study M-ideals in A by means of the corresponding L-ideals in A^*, which in turn are studied by geometric and analytic properties of the closed unit ball K in A^*.

Although we have an isometric complex-linear representation of the given function space as the space of all complex-valued w^*-continuous linear functions on K, it turns out that the smaller compact, convex set $Z = \text{conv}(S_A \cup -iS_A)$, where S_A denotes the state space of A, will contain enough structure to determine the L-ideals. The set Z was first studied by Asimow [4]. Note also that the problems which always arise in the presence of complex orthogonal measures can, to a certain extent, be given a geometric treatment when we consider the compact, convex set Z (Theorem 2.4).

Another useful tool in this context is the possibility of representing complex linear functionals by complex boundary measures of the same norm, as was recently proved by Hustad [11].

Specializing to uniform algebras we characterize the M-summands (see [2, §5]), and we conclude by pointing out that the structure-topology of Alfsen and Effros [2, §6] coincides with the symmetric facial topology studied by Ellis in [7]. This result yields a description of the structure space, Prim A (see [2, §6]), in terms of concepts more familiar to function algebraists. Specifically, Prim A is (homeomorphic to) the Choquet-boundary of X endowed with the p-set topology.

The author wants to express his gratitude to E. Effros for Theorem 1.2 and to E. Alfsen for helpful comments.

1. Preliminaries and notation

Let W denote a real Banach space. Following [2, §3] we define an *L-projection* e on W to be a linear map of W into itself such that,

i) $e^2 = e$

ii) $\| p \| = \| e(p) \| + \| p - e(p) \| \qquad \forall p \in W$

and we define the range of an L-projection to be an *L-ideal* in W.

To every L-ideal $N = eW$, there is associated a *complementary L-ideal* $N' = (I - e)W$. See [2, §3].

We say that a closed subspace J of a real Banach space V is an *M-ideal* if the polar of J is an L-ideal in $W = V^*$. Also, we define a linear map e of V into itself to be an *M-projection* if

i) $e^2 = e$

ii) $\| v \| = \max\{\| e(v) \|, \| v - e(v) \|\} \qquad \forall v \in V$

and we define a subspace of V to be an *M-summand* if it is the range of an M-projection. It follows from [2, Corollary 5.16] that M-summands are M-ideals.

LEMMA 1.2. *Let N be an L-ideal in a real Banach-space W, and let e be the corresponding L-projection. If T is an isometry of W onto itself, then TN is an L-ideal and the corresponding L-projection e_T is given by*

$$e_T = TeT^{-1}. \tag{1.1}$$

Also

$$(TN)' = T(N').$$

PROOF. Straightforward verification.

If V is a complex Banach space, then we shall denote by V_r the *subordinate real space*, having the same vectors but equipped with real scalars only. By an elementary theorem [12, §6], it follows that there is a natural isometry ϕ of (V^*) onto $(V_r)^*$, defined by

$$\phi(p)(v) = \operatorname{Re} p(v) \qquad v \in V. \tag{1.2}$$

THEOREM 1.2. (Effros) *Let W be a complex Banach space with subordinate real space W_r. If N is an L-ideal in W_r, then N is a complex linear subspace of W.*

PROOF. It suffices to prove that $ip \in N$ for all $p \in N$. Let $p \in N$ and consider

$$q = p - e_T p,$$

where T is the isometry $T(p) = ip/\forall p \in W$ and e_T is defined as in (1.1). Then

$$q = e(p) - e_T e(p) = e(p - e_T(p)) \in N$$

since L-projections commute [5, Lemma 2.2.]. Also we have

$$iq = i(I - e_T)(p) \in i(T(N')) = N'.$$

Thus

$$\sqrt{2} \| q \| = \| q + iq \| = \| q \| + \| iq \| = 2 \| q \| .$$

Hence $q = 0$ and $ip \in N$.

COROLLARY 1.3. *Let V be a complex Banach space with subordinate real space V_r. If J is an M-ideal in V_r, then J is a complex linear subspace of V.*

PROOF. It suffices to prove that if $v \in J$ then $iv \in J$. Assume for contradiction that there exists $v_0 \in J$ such that $iv_0 \notin J$. By the Hahn-Banach theorem there exists $p \in V^*$ such that

$$\mathrm{Re}\, p(iv_0) > 0, \ \mathrm{Re}\, p(v) = 0 \qquad \forall v \in J.$$

Let $\phi: (V^*)_r \to (V_r)^*$ and $T: (V^*)_r \to (V^*)_r$ as above. If J_r^0 denotes the polar of J in $(V_r)^*$ then $\phi^{-1}(J_r^0)$ is an L-ideal in $(V^*)_r$ and hence a complex linear subspace of V^*. Thus $T\phi^{-1}(J_r^0) \subseteq \phi^{-1}(J_r^0)$ and moreover,

$$J_r^0 = (\phi T \phi^{-1})(J_r^0)$$

Since $\mathrm{Re}\, p \in J_r^0$ we shall have $(\phi T \phi^{-1})(\mathrm{Re}\, p) \in J_r^0$. Thus

$$0 = (\phi T \phi^{-1})(\mathrm{Re}\, p)(v_0) = \mathrm{Re}(ip)(v_0) = \mathrm{Re}\, p(iv_0)$$

and we have obtained a contradiction.

The above results justify the use of the terms L- *and* M-*ideals* for *complex Banach spaces* to denote L- and M-ideals in the subordinate real spaces.

Let V be a complex Banach space, $W = V^*$, and K the closed unit ball of W. If N is a w^*-closed L-ideal in W with corresponding L-projection e, then it follows from [2, Corollary 4.2] that, for a given $v \in V$ considered as a complex linear function in W, one has $v \circ e$ is Borel and

$$(v \circ e)(p) = \int_K (v \circ e)\, d\mu \qquad \forall p \in K, \ \forall \mu \in M_p^+(K) \tag{1.3}$$

and

$$(v \circ e)(p) = \int_{N \cap K} v \, d\mu \qquad \forall p \in K, \ \forall \mu \in M_p^+(\partial_e K) \tag{1.4}$$

where $M_p^+(K)$ denotes the set of all probability measures on K with barycenter p, and $M_p^+(\partial_e K)$ the set of all measures in $M_p^+(K)$ which are maximal in Choquet's ordering (boundary measures).

2. M-ideals in complex function spaces

In this section, X shall denote a compact Hausdorff space and A a closed, linear subspace of $\mathscr{C}_{\mathbb{C}}(X)$, which separates the points of X and contains the constant functions. The *state space* of A, i.e.,

$$S_A = \{p \in A^* \mid p(\mathbf{1}) = \| p \| = 1\}$$

is a w^*-closed face of the closed unit ball K of A^*. We shall assume that K is endowed with w^*-topology.

Since A separates the points of X, we have a homeomorphic embedding Φ of X into S_A, defined by

$$\Phi(x)(a) = a(x) \qquad \forall a \in A. \tag{2.1}$$

We use θa to denote the function on A^* defined by

$$\theta a(p) = \operatorname{Re} p(a) \qquad \forall p \in A^*. \tag{2.2}$$

For convenience we shall use the same symbol θa to denote the *restriction* of this function to various compact, convex subsets of A^*.

An enlargement of S_A, which was introduced by Asimow, is the following set

$$Z = \operatorname{conv}(S_A \cup - iS_A). \tag{2.3}$$

Appealing to [4, Proposition 1], the embedding $a \to \theta a$ is a bicontinuous real linear isomorphism of A onto the space $A(Z)$ of all real-valued w^*-continuous affine functions on Z.

We shall denote by $M_1^+(S_A)$ (respectively $M_1^+(Z)$) the w^*-compact convex set of probability measures on S_A (respectively Z). The set of extreme points of S_A (respectively Z, K) will be denoted by $\partial_e S_A$ (respectively $\partial_e Z$, $\partial_e K$) and the Choquet boundary of X with respect to A is defined as the set

$$\partial_A X = \{x \in X \mid \Phi(x) \in \partial_e S_A\}.$$

It follows from [12, p. 38] that $\partial_e S_A \subseteq \Phi(X)$. Moreover,

$$\partial_e K = \{\lambda\Phi(x) \mid |\lambda| = 1, \ x \in \partial_A X\}.$$

See [6, p. 441].

Also we agree to write $M_p^+(S_A)$ (respectively $M_z^+(Z)$) for the w^*-compact convex set of probability measures on S_A (respectively Z) which has barycenter $p \in S_A$ (respectively $z \in Z$). By $M_p^+(\partial_e S_A)$ (respectively $M_z^+(\partial_e Z)$) we denote the maximal representing measures for p (respectively z) (boundary measures).

A *real measure* μ on S_A (respectively Z, K) is said to be a *boundary measure* on S_A (respectively Z, K) if the total variation $|\mu|$ is a maximal element in the Choquet ordering, and we denote them by $M(\partial_e S_A)$ (respectively $M(\partial_e Z)$, $M(\partial_e K)$).

Finally we denote by $M(\partial_A X)$ those *complex* measures μ on X for which the direct image measure $\Phi(|\mu|)$ on S_A is an element of $M(\partial_e S_A)$.

It is well-known (see e.g. [1, Proposition I.4.6]) that boundary measures are supported by the closure of the extreme boundary.

As mentioned we shall study M-ideals in A by considering the corresponding

L-ideals in A^*. Let N be a w^*-closed L-ideal in A^* with corresponding L-projection e.

LEMMA 2.1. *Let $p \in S_A$. Then*

$$e(p) \in \mathrm{conv}(\{0\} \cup S_A).$$

PROOF. Let $p \in S_A$ and decompose $p = q + r$ where $q = e(p)$ and $r = (I - e)(p)$. If $q = 0$ or $r = 0$ there is nothing to prove. Otherwise

$$p = \| q \| \left(\frac{q}{\| q \|} \right) + \| r \| \left(\frac{r}{\| r \|} \right)$$

is a convex combination of points in K. Since S_A is a face of K we obtain $q / \| q \| \in S_A$. Hence

$$e(p) = q \in \mathrm{conv}(\{0\} \cup S_A).$$

COROLLARY 2.2. *Let $z \in Z$, then*

$$(I - e)(z) \in \mathrm{conv}(\{0\} \cup Z).$$

PROOF. Since $I - e$ is an L-projection the corollary follows immediately from Lemma 2.1 and the definition of Z.

If Q is a closed face of a compact, convex set H, then the *complementary face* Q' is the union of all faces disjoint from Q. Q is said to be a *split face* of H if Q' is convex and each point in $K \setminus (Q \cup Q')$ can be expressed uniquely as a convex combination of a point in Q and a point in Q', (cf [1, p. 133]).

According to [1, Theorem II.6.12], we have that for a closed face Q of H the following statements are equivalent:

(i) Q is a split face.

(ii) If $\mu \in M(\partial_e H)$ annihilates all continuous affine functions, then $\mu | Q$ has the same property.

THEOREM 2.3. *Let N be a w^*-closed L-ideal of A^* and let $F = N \cap Z$. Then F is a split face of Z with complementary face $F' = N' \cap Z$.*

PROOF. To see that F is a face of Z we consider a convex combination

$$\lambda z_1 + (1 - \lambda) z_2 \in F,$$

where $z_1, z_2 \in Z$ and $0 < \lambda < 1$. From Corollary 2.2, we have $(I - e)(z_i) \in \mathrm{conv}(\{0\} \cup Z)$ for $i = 1, 2$. Moreover,

$$0 = \lambda (I - e)(z_1) + (1 - \lambda)(I - e)(z_2). \tag{2.4}$$

Since $0 \notin Z$, 0 is an extreme point of $\operatorname{conv}(\{0\} \cup Z)$ and hence from (2.4)

$$(I - e)(z_1) = 0 = (I - e)(z_2).$$

Consequently $z_i \in F$ for $i = 1,2$ and F is a face of Z.

Let $z \in F'$ and $\mu \in M_z^+(\partial_e Z)$. Then $\mu(F) = 0$ [9, Lemma 2.11]. Moreover, the Milman theorem implies that $\partial_e Z \subseteq (S_A \cup - i S_A)$ and hence $\operatorname{Supp}(\mu) \subseteq (S_A \cup - iS_A)$. Since these two sets are faces of K we may consider μ as a boundary measure on K.

According to (1.4), we also have

$$(\theta a \circ e)(z) = \int_F \theta a \, d\mu \qquad \forall a \in A,$$

where e is the L-projection corresponding to N. Thus $e(z) = 0$, which in turn implies $z \in N' \cap Z$.

Conversely, assume $z \in N' \cap Z$. Decompose

$$z = \lambda p_1 + (1 - \lambda)p_2$$

where $p_1 \in F$, $p_2 \in F'$ and $0 \leqq \lambda \leqq 1$. Hence

$$z - (1 - \lambda)p_2 = \lambda p_1 \in N \cap N' = \{0\},$$

and so $z = p_2 \in F'$. Thus we have proved that $F' = N' \cap Z$. In particular, F' is convex.

From the above results we may establish the splitting property by proving

$$\mu \in A(Z)^{\perp} \cap M(\partial_e Z) \Rightarrow \mu|_F \in A(Z)^{\perp}.$$

To this end we consider $\mu \in A(Z)^{\perp} \cap M(\partial_e Z)$. As before $\mu \in M(\partial_e K)$, and also

$$\int_K \theta a \, d\mu = \int_Z \theta a \, d\mu = 0 \qquad \forall a \in A,$$

i.e., $\mu \in A_0(K)^{\perp} \cap M(\partial_e K)$, where $A_0(K)$ is the space of all real-valued w^*-continuous linear functions on K. By virtue of [2, Theorem 4.5] $\mu|_F \in A_0(K)^{\perp}$, or equivalently $\mu|_F \in A(Z)^{\perp}$. *q.e.d.*

REMARK. Under the hypothesis of Theorem 2.3 we have

$$F = \operatorname{conv}((F \cap S_A) \cup - i(F \cap S_A)).$$

Following Ellis [7] we shall say that a subset of Z of the form

$$\operatorname{conv}(C \cup - iC), \quad C \subseteq S_A$$

is *symmetric*. Let F be a closed face of S_A, and put

$$E = \Phi^{-1}(F \cap \Phi(X)). \tag{2.5}$$

Then $F = \overline{\mathrm{conv}}(\Phi(E))$ and $F \cap \Phi(X) = \Phi(E)$.

THEOREM 2.4. *Let F be a closed face of S_A and let E be as in* (2.5). *Then the following statements are equivalent*:

(i) $S_F = \mathrm{conv}(F \cup -iF)$ *is a split face of Z.*

(ii) $\mu \in A^{\perp} \cap M(\partial_A X) \Rightarrow \mu|_E \in A^{\perp}$.

PROOF. Assume S_F is a split face and let $\mu \in A^{\perp} \cap M(\partial_A X)$. Let $\sigma = \Phi\mu$. Then σ is a complex boundary measure on S_A. If v is a real or complex measure on K, then we denote by $r(v)$ the resultant of v, i.e., the unique point in A^* for which

$$a(r(v)) = \int_K a dv \qquad \forall a \in A. \tag{2.6}$$

Since $\mu \in A^{\perp}$ we have $r(\sigma) = 0$. Rewrite σ as

$$\sigma = \sigma_1 + i\sigma_2 \tag{2.7}$$

where σ_i is a real boundary measure on K for $i = 1,2$. Define $\psi: S_A \to -iS_A$ by

$$\psi(p) = -ip \qquad \forall p \in S_A. \tag{2.8}$$

The measure

$$\sigma' = \sigma_1 - \psi(\sigma_2) \tag{2.9}$$

is a real boundary measure on Z with $r(\sigma') = 0$, i.e., $\sigma' \in A(Z)^{\perp} \cap M(\partial_e Z)$. Since S_F is a split face of Z, we shall have $\sigma'|_{S_F} \in A(Z)^{\perp}$ and hence

$$\begin{aligned} 0 = r(\sigma'|_{S_F}) &= r(\sigma_1|_F) - r(\psi(\sigma_2)|_{\psi(F)}) \\ &= r(\sigma_1|_F) + ir(\sigma_2|_F) = r(\sigma|_F) \end{aligned}$$

or equivalently $\mu|_E \in A^{\perp}$.

Assume conversely that E satisfies (ii). First we prove that S_F is a face of Z. Let $z \in S_F$ and $\mu \in M_z^+(\partial_e Z)$. Then we have to prove that $\mathrm{Supp}(\mu) \subseteq S_F$. Since $z \in S_F$ we may write z as a convex combination

$$z = \lambda p_1 + (1-\lambda)(-ip_2), \qquad p_i \in F \quad i = 1,2.$$

Choose $\sigma_i \in M_{p_i}^+(\partial_e S_A)$ for $i = 1,2$. Then $\sigma_i(F) = 1$ since F is a face. Since $\mu \in M_z^+(\partial_e Z)$ we may write μ as

$$\mu = \mu_1 + \mu_2, \tag{2.10}$$

where $\mathrm{Supp}(\mu_1) \subseteq S_A$, $\mathrm{Supp}(\mu_2) \subseteq -iS_A$.

Consider the measure

$$\nu = \lambda\sigma_1 + (1-\lambda)(-i\sigma_2) - (\mu_1 - i\psi^{-1}(\mu_2)). \tag{2.11}$$

Then ν is a complex boundary measure on S_A with $r(\nu) = 0$. From (ii) it follows that $r(\nu|_F) = 0$. Specifically $\nu(F) = 0$ and hence from (2.11)

$$0 = \lambda + (1-\lambda)(-i) - \mu_1(F) + i\mu_2(-iF),$$

i.e., $\mu_1(F) = \lambda$ and $\mu_2(-iF) = 1 - \lambda$ and hence $\mu(S_F) = 1$.

To prove that S_F is a split face we let $\mu \in A(Z)^\perp \cap M(\partial_e Z)$. As in (2.10) we write μ as $\mu = \mu_1 + \mu_2$ and define

$$\mu' = \mu_1 - i\psi^{-1}(\mu_2).$$

Then $r(\mu') = 0$ and as above $r(\mu'|_F) = 0$. Hence

$$0 = r(\mu_1|_F) - ir(\psi^{-1}(\mu_2)|_F) = r(\mu|_{S_F}),$$

i.e., $\mu|_{S_F} \in A(Z)^\perp$ and the theorem is proved.

THEOREM 2.5. *Let F be a closed face of S_A for which $S_F = \text{conv}(F \cup -iF)$ is a split face of Z. Then*

$$N = \text{lin}_{\mathbb{C}} F$$

is a w^*-closed L-ideal in A^*.

PROOF. Since S_F is a split face, N may be considered as a w^*-closed real linear subspace of $A(Z)^*$ and from the connection between A and $A(Z)$ (see section 1) it follows that N is w^*-closed in A^*.

According to Theorem 2.4 the following definition is legitimate,

$$e(p)(a) = \int_E a\,d\mu \qquad \forall a \in A,$$

where E is as in (2.5) and μ is a maximal complex measure representing the point $p \in A^*$. Clearly $e(A^*) \subseteq N$. Let $p \in N$, i.e.,

$$p = \lambda_1 p_1 + \lambda_2(-p_2) + \lambda_3(ip_3) + \lambda_4(-ip_4)$$

where $p_i \in F$ and $\lambda_i \geqq 0$ for $i = 1,2,3,4$.

Choose measures $\sigma_i \in M^+_{p_i}(\partial_e S_A)$ for $i = 1,2,3,4$. Then $\text{Supp}(\sigma_i) \subseteq \Phi(E)$ since F is a face of S_A. Define $\mu_i = \Phi^{-1}\sigma_i$ for $i = 1,2,3,4$ and

$$\mu = \lambda_1\mu_1 - \lambda_2\mu_2 + i\lambda_3\mu_3 - i\lambda_4\mu_4.$$

Now μ is a complex representing measure for p and $\text{Supp}(\mu) \subseteq E$, i.e.,

$$e(p) = p.$$

To prove that e is an L-projection, we shall need the fact that we may represent $p \in A^*$ by a measure $\mu \in M(\partial_A X)$ such that $\| p \| = \|\mu\|$. This follows by a slight modification of a theorem of Hustad [11] (cf. [10]).

Having chosen $\mu \in M(\partial_A X)$ representing $p \in A^*$ with $\| p \| = \| \mu \|$, we have

$$\| p \| \leqq \| e(p) \| + \| p - e(p) \| \leqq \| \mu \|_E + \| \mu \|_{X/E} = \| \mu \| = \| p \|,$$

which implies

$$\| p \| = \| e(p) \| + \| p - e(p) \| \qquad \forall p \in A^*,$$

i.e., e is an L-projection with range N.

A compact subset $E \subseteq X$ is said to be *A-convex* if it satisfies:

$$E = \{x \in X \mid \ |a(x)| \leqq \| a \|_E \qquad \forall a \in A\}.$$

If F is a closed face of S_A such that $S_F = \operatorname{conv}(F \cup -iF)$ is a split face of Z, then the set $E = \Phi^{-1}(F \cap \Phi(X))$ is A-convex and has the following properties:

(i) $\mu \in M_1^+(\partial_A X)$, $\nu \in M_1^+(E)$, $\mu - \nu \in A^\perp \Rightarrow \operatorname{Supp}(\mu) \subseteq E$.

(ii) $\mu \in A^\perp \cap M(\partial_A X) \Rightarrow \mu|_E \in A^\perp$.

If an A-convex subset E of X satisfies (i) and (ii) then we say that E is an *M-set*.

If $E \subseteq X$ is a compact subset then we denote by S_E the following subset of S_A,

$$S_E = \overline{\operatorname{conv}}(\Phi(E)). \tag{2.12}$$

Clearly, if E is an M-set, S_E is a closed face of S_A and $S_E \cap \Phi(X) = \Phi(E)$.

Moreover,

COROLLARY 2.6. *Let E be an M-set of X. Then*

$$N = \operatorname{lin}_{\mathbf{C}} S_E$$

is a w^*-closed L-ideal of A^*.

PROOF. Theorems 2.4 and 2.5.

COROLLARY 2.7. *Let E be an A-convex subset of X. Then the following statements are equivalent:*

(i) *E is an M-set.*

(ii) $\operatorname{conv}(S_E \cup -iS_E)$ *is a split face of Z.*

(iii) $\operatorname{conv}(S_E \cup -iS_E)$ *is a face of Z and $N = \operatorname{lin}_{\mathbf{C}} S_E$ is a w^*-closed L-ideal.*

PROOF. Theorems 2.3, 2.4 and 2.5.

REMARK. Thus we see that there is a one-to-one correspondence between the w^*-closed L-ideals of A^* and the closed symmetric split faces of Z.

That not all split faces of Z are symmetric is a consequence of the following observation:

A closed face F of S_A is a split face of Z if and only if the following condition is satisfied:

$$\mu \in A^{\perp} \cap M(\partial_A X) \Rightarrow \begin{cases} \mu_1 |_E \in A^{\perp} \\ \mu_2 |_E \in A^{\perp} \end{cases}$$

where $\mu = \mu_1 + i\mu_2$ and E is as in (2.5).

REMARK. See [4] and [7] for similar results.

Turning to the M-ideals in A we now have the following

THEOREM 2.8. *Let J be a closed subspace of A. Then the following statements are equivalent:*

(i) *J is an M-ideal.*

(ii) $J = \{a \in A \mid a \equiv 0 \text{ on } E\}$,

where E is an M-set of X.

PROOF. Assume J is an M-ideal of A. Then $J^0 \cap Z$ is a split face of Z since J^0 is an L-ideal. Moreover, we claim that

$$J^0 = \mathrm{lin}_{\mathbb{C}}(J^0 \cap S_A).$$

Trivially, $\mathrm{lin}_{\mathbb{C}}(J^0 \cap S_A) \subseteq J^0$. If $p \in \partial_e(J^0 \cap K)$ then

$$p \in \partial_e(J^0 \cap K) = J^0 \cap \partial_e K.$$

Hence

$$p = \lambda q, \quad |\lambda| = 1, \quad q \in \partial_e S_A.$$

Thus

$$q = \lambda^{-1} p \in J^0 \cap S_A$$

such that

$$p \in \mathrm{lin}_{\mathbb{C}}(J^0 \cap S_A).$$

It follows from Theorem 2.5 that $\mathrm{lin}_{\mathbb{C}}(J^0 \cap S_A)$ is w^*-closed and hence

$$\overline{\mathrm{conv}}(\partial_e(J^0 \cap K)) \subseteq \mathrm{lin}_{\mathbb{C}}(J^0 \cap S_A).$$

This in turn implies

$$J^0 = \mathrm{lin}_{\mathbb{C}}(J^0 \cap S_A).$$

Equivalently

$$J^0 = \overline{\mathrm{lin}_{\mathbb{C}}(\Phi(E))}^{W^*},$$

where $E = \Phi^{-1}(J^0 \cap \Phi(X))$.

Thus we see that

$$J = \{a \in A \mid a \equiv 0 \text{ on } E\},$$

and clearly E is an M-set.

Conversely, if J is of the form

$$J = \{a \in A \mid a \equiv 0 \text{ on } E\}.$$

where E is an M-set, then $J^0 = \mathrm{lin}_{\mathbb{C}} S_E$ is an L-ideal according to Corollary 2.6.

3. The uniform algebra case

In this section we make the further assumption that A is a uniform algebra [8].

A *peak set* E for A is a subset of X for which there exists a function $a \in A$ such that

$$a(x) = 1 \;\; \forall x \in E, \qquad |a(x)| < 1 \qquad \forall x \in X \setminus E.$$

A *p-set* (generalized peak set) for A is an intersection of peak-sets for A. If X is metrizable then every p-set is a peak set [8, §12].

It follows from [8, Theorem 12.7] that the following are equivalent for a compact subset E of X:

(i) E is a p-set.

(ii) $\mu \in A^{\perp} \Rightarrow \mu|_E \in A^{\perp}$.

Clearly, p-sets are M-sets. Moreover, since M-sets are A-convex it follows by a slight modification of [3, Theorem 7.4] that M-sets are p-sets, i.e., we may state

THEOREM 3.1. *Let A be a uniform algebra and J a closed subspace of A. Then the following statements are equivalent:*

(i) J is an M-ideal.

(ii) $J = \{a \in A \mid a \equiv 0$ on $E\}$,

where E is a p-set for A.

Turning to the M-summands of A we have

THEOREM 3.2. *Let J be a closed subspace of A. Then the following statements are equivalent:*

(i) *J is an M-summand.*

(ii) $J = \{a \in A \mid a \equiv 0 \text{ on } E\}$,

where E is an open-closed p-set for A.

PROOF. Trivially (ii) $\Rightarrow$ (i) by virtue of Theorem 3.1.

Conversely, assume J is an M-summand. Then

$$J = \{a \in A \mid a \equiv 0 \text{ on } E\},$$

where E is a p-set for A. To prove that E is open it suffices to prove that

$$\{x \in X \mid e(\mathbf{1})(x) = 1\} = X \setminus E$$

where e is the M-projection corresponding to J. Clearly

$$\{x \in X \mid e(\mathbf{1})(x) = 1\} \subseteq X \setminus E.$$

Let $x \notin E$, and μ a maximal measure on X representing x. Then $(\mu - \varepsilon_x) \in A^{\perp}$ and hence $\mu(E) = 0$.

Moreover, if e^* denotes the adjoint of e, then $(eA)^0 = (I - e^*)A^*$ and hence

$$\mathbf{1} \circ (I - e^*)(\Phi(x)) = \int_E \mathbf{1}\, d\mu = 0.$$

Thus

$$0 = (I - e^*)(\Phi(x))(\mathbf{1}) = 1 - e(\mathbf{1})(x)$$

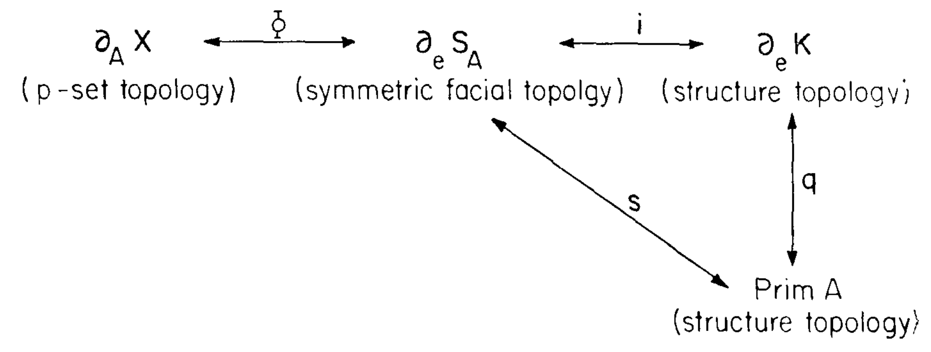
and we are done. (cf. [2, Corollary 5.16]).

Finally we point out that since every point $x \in \partial_A X$ is a p-set for A and

$$J_x = \{a \in A \mid a(x) = 0\}$$

is the largest M-ideal contained in the kernel of $\Phi(x)$, then the *structure-topology* [2, §6] on $\partial_e K$ restricted to $\partial_e S_A$ coincides with the *symmetric facial topology* studied by Ellis in [7]. This follows from Theorems 2.3 to 2.5.

Moreover, this topology coincides with the well known *p-set topology*. Specifically, if $p \in \partial_e K$, then there exists a unique point $x_p \in \partial_A X$ and $\lambda_p \in \{z \in C \mid |z| = 1\}$ such that $p = \lambda_p \Phi(x_p)$ and hence the largest M-ideal contained in the kernel of p is J_{x_p}. The above can be summed up in the following diagram:



where all the maps are continuous, q is open, and Φ and s are homeomorphisms.

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